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COMPARISON OF ADMISSIBILITY CONDITIONS FOR CYCLOTOMIC BIRMAN–WENZL–MURAKAMI ALGEBRAS

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ABSTRACT. We show the equivalence of admissibility conditions proposed by Wilcox and Yu [11] and by Rui and Xu [9] for the parameters of cyclotomic BMW algebras.

1. INTRODUCTION

Cyclotomic Birman–Wenzl–Murakami (BMW) algebras are BMW analogues of cyclotomic Hecke algebras [2, 1]. They were defined by Häring–Oldenburg in [7] and have recently been studied by three groups of mathematicians: Goodman and Hauschild–Mosley [4, 5, 6, 3], Rui, Xu, and Si [9, 8], and Wilcox and Yu [11, 12, 10, 13].

A peculiar feature of these algebras is that it is necessary to impose "admissibility" conditions on the parameters entering into the definition of the algebras in order to obtain a satisfactory theory. There is no one obvious best set of conditions, and the different groups studying these algebras have proposed different admissibility conditions and have chosen slightly different settings for their work.

Under their various admissibility hypotheses on the ground ring, the several groups of mathematicians mentioned above have obtained similar results for the cyclotomic BMW algebras, namely freeness and cellularity. In addition, Goodman & Hauschild–Mosley and Wilcox & Yu have shown that the algebras can be realized as algebras of tangles, while Rui et. al. have obtained additional representation theoretic results, for example, classification of simple modules and semisimplicity criteria. However, it has been difficult to compare the results of the different investigations because of the different settings.

The purpose of this note is to show that the admissibility condition proposed by Rui and Xu [9] is equivalent to the condition proposed by Wilcox and Yu [11]. As a result, one has a consensus setting for the study of cyclotomic BMW algebras.

Further background on cyclotomic BMW algebras, motivation for the study of these algebras, relations to other mathematical topics (quantum groups, knot theory), and further literature citations can be found in [5] and in the other papers cited above.

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2. Definitions

In general we use the definitions and notation from [6].

Definition 2.1. Fix an integer $r \geq 1$. A *ground ring* S is a commutative unital ring with parameters ρ, q, δ_j $(j \geq 0)$, and $u_1, \dots, u_r$, with ρ, q, and $u_1, \dots, u_r$ invertible, and with

$$\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1). \tag{2.1}$$

Definition 2.2. Let S be a ground ring with parameters ρ, q, δ_j $(j \geq 0)$, and $u_1, \dots, u_r$. The *cyclotomic BMW algebra* $W_{n,S,r}(u_1, \dots, u_r)$ is the unital S–algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i $(1 \leq i \leq n-1)$ and relations:

1. (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
2. (Idempotent relation) $e_i^2 = \delta_0 e_i$.
3. (Affine braid relations)
   - (a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i - j| \geq 2$.
   - (b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geq 2$.
4. (Commutation relations)
   - (a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.
   - (b) $y_1 e_j = e_j y_1$ if $j \geq 2$.
5. (Affine tangle relations)
   - (a) $e_i e_{i\pm 1} e_i = e_i$,
   - (b) $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$ and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.
   - (c) For $j \geq 1$, $e_1 y_1^j e_1 = \delta_j e_1$.
6. (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
7. (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i\pm 1} e_i = \rho e_i$.
8. (Unwrapping relation) $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.
9. (Cyclotomic relation) $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$.

Thus, a cyclotomic BMW algebra is the quotient of the affine BMW algebra [7, 4], by the cyclotomic relation $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$. We recall from [4] that the affine BMW algebra is isomorphic to an algebra of framed affine tangles, modulo Kauffman skein relations. Assuming admissible parameters, it has been shown that the cyclotomic BMW algebras are also isomorphic to tangle algebras [6, 10, 13].

Lemma 2.3. *For $j \geq 1$, there exist elements $\delta_{-j} \in \mathbb{Z}[\rho^{\pm 1}, q^{\pm 1}, \delta_0, \dots, \delta_j]$ such that $e_1 y_1^{-j} e_1 = \delta_{-j} e_1$. Moreover, the elements δ_{-j} are determined by the recursion relation:*

$$\begin{aligned} \delta_{-1} &= \rho^{-2} \delta_1 \\ \delta_{-j} &= \rho^{-2} \delta_j + (q^{-1} - q)\rho^{-1} \sum_{k=1}^{j-1} (\delta_k \delta_{k-j} - \delta_{2k-j}) \quad (j \geq 2). \end{aligned} \tag{2.2}$$

Proof. Follows from [4], Corollary 3.13, and [5], Lemma 2.6; or [9], Lemma 2.17. □

We consider what are the appropriate morphisms between ground rings for cyclotomic BMW algebras. The obvious notion would be that of a ring homomorphism taking parameters to parameters; that is, if S is a ground ring with parameters ρ, q, etc., and S' another ground ring with parameters ρ', q', etc., then a morphism $\varphi : S \to S'$ would be required to map $\rho \mapsto \rho'$, $q \mapsto q'$, etc.

However, it is better to require less, for the following reason: The parameter q enters into the cyclotomic BMW relations only in the expression $q^{-1} - q$, and the transformation $q \mapsto -q^{-1}$ leaves this expression invariant. Moreover, the transformation $g_i \mapsto -g_i$, $\rho \mapsto -\rho$, $q \mapsto -q$ (with all other generators and parameters unchanged) leaves the cyclotomic BMW relations unchanged.

Taking this into account, we arrive at the following notion:

Definition 2.4. Let S be a ground ring with parameters ρ, q, δ_j ($j \geq 0$), and $u_1, \dots, u_r$. Let S' be another ground ring with parameters ρ', q', etc.

A unital ring homomorphism $\varphi : S \to S'$ is a *morphism of ground rings* if it maps

$$\begin{cases} \rho \mapsto \rho', \text{ and} \\ q \mapsto q' \text{ or } q \mapsto -q'^{-1}, \end{cases}$$

or

$$\begin{cases} \rho \mapsto -\rho', \text{ and} \\ q \mapsto -q' \text{ or } q \mapsto q'^{-1}, \end{cases}$$

and strictly preserves all other parameters.

Suppose there is a morphism of ground rings $\psi : S \to S'$. Then ψ extends to a homomorphism from $W_{n,S,r}$ to $W_{n,S',r}$. Moreover, $W_{n,S,r} \otimes_S S' \cong W_{n,S',r}$ as S'–algebras. These statements are discussed in [6], Section 2.4.

3. Admissibility conditions

The following *weak admissibility* condition is a minimal condition on the parameters to obtain a non–trivial algebra; in the absence of weak admissibility, the generator e_1 is a torsion element over the ground ring; if S is a field, then $e_1 = 0$, and the cyclotomic BMW algebra reduces to a specialization of the cyclotomic Hecke algebra. See the remarks preceding Definition 2.14 in [6].

In the following definition, a_j denotes the signed elementary symmetric function in $u_1, \dots, u_r$, namely, $a_j = (-1)^{r-j} \varepsilon_{r-j}(u_1, \dots, u_r)$.

Definition 3.1. Let S be a ground ring with parameters ρ, q, δ_j, $j \geq 0$, and $u_1, \dots, u_r$. We say that the parameters are *weakly admissible* (or that the ring S is weakly admissible) if the following relation holds:

$$\sum_{k=0}^{r} a_k \delta_{k+a} = 0,$$

for $a \in \mathbb{Z}$, where for $j \geq 1$, δ_{-j} is defined by the recursive relations of Lemma 2.3.

In order to obtain substantial results on the cyclotomic BMW algebras, it appears necessary to impose a condition on the ground ring that is stronger than weak admissibility. Two conditions have been proposed, one by Wicox and Yu, and another by Rui and Xu.

First we consider the admissibility condition of Wilcox and Yu. Consider a ground ring S with parameters ρ, q, δ_j $(j \ge 0)$ and $u_1, \dots, u_r$. Let W_2 denote the cyclotomic BMW algebra $W_2 = W_{2,S,r}(u_1, \dots, u_r)$.

Lemma 3.2. *The left ideal $W_2 e_1$ in W_2 is equal to the S–span of $\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\}$.*

Theorem 3.3 (Wilcox & Yu, [11])**.** *Let S be a ground ring with parameters ρ, q, δ_j $(j \ge 0)$ and $u_1, \dots, u_r$. Assume that $(q - q^{-1})$ is not a zero–divisor in S. The following conditions are equivalent:*

(1) *$\{e_1, y_1 e_1, \dots, y_1^{r-1} e_1\} \subseteq W_2$ is linearly independent over S.*
(2) *The parameters satisfy the following relations:*

$$\begin{aligned} &\rho(a_\ell - a_{r-\ell}/a_0) + \\ &(q - q^{-1})\left[\sum_{j=1}^{r-\ell} a_{j+\ell}\delta_j - \sum_{j=\max(\ell+1,\lceil r/2\rceil)}^{\lfloor(\ell+r)/2\rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2\rceil}^{\min(\ell,\lceil r/2\rceil-1)} a_{2j-\ell}\right] = 0, \\ &\quad \text{for } 1 \le \ell \le r-1, \end{aligned} \tag{3.1}$$

$$\rho^{-1} a_0 - \rho a_0^{-1} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (q - q^{-1}) & \text{if } r \text{ is even,} \end{cases} \tag{3.2}$$

and

$$\delta_a = -\sum_{j=0}^{r-1} a_j \delta_{a-r+j} \quad \text{for } a \ge r. \tag{3.3}$$

(3) *S is weakly admissible, and W_2 admits a module M with an S–basis $\{v_0, y_1 v_0, \dots, y_1^{r-1} v_0\}$ such that $e_1(y_1^j v_0) = \delta_j v_0$ for $0 \le j \le r-1$, $g_1 v_0 = \rho^{-1} v_0$, and $y_2 y_1^j v_0 = y_1^{j-1} v_0$.*

Definition 3.4 (Wilcox and Yu, [11])**.** Let S be a ground ring with parameters ρ, q, δ_j $(j \ge 0)$ and $u_1, \dots, u_r$. Assume that $(q - q^{-1})$ is not a zero–divisor in S. One says that S is *admissible* (or that the parameters are *admissible*) if the equivalent conditions of Theorem 3.3 hold.

Remark 3.5.

(1) In later work, Wilcox and Yu considered a more subtle version of their admissibility condition that is also valid if $q - q^{-1}$ is a zero–divisor.
(2) If R is an integral ground ring and Equation (3.2) holds, then $\rho = \pm a_0$ if r is odd, and $\rho \in \{q^{-1} a_0, -q a_0\}$ if r is even.

Next we discuss the admissiblity condition of Rui and Xu [9], called u*-admissibility*. In [9], ground rings are assumed to be integral domains, and it is assumed that $q-q^{-1}$ is invertible. Since we do not want to specialize to this situation immediately, the form in which we describe u–admissibility will be a little different from that in [9].

The definition of u–admissibility is based on a heuristic involving linear independence of $\{e_1, y_1e_1, \dots, y_1^{r-1}e_1\} \subseteq W_2$, under additional assumptions on $u_1, \dots, u_r$. Suppose that F is a field and $u_1, \dots, u_r$ are *distinct* invertible elements of F with $u_iu_j \neq 1$ for all i, j. Moreover, suppose ρ and q are non-zero elements of F with $q - q^{-1} \neq 0$. Define quantities γ_j $(1 \le j \le r)$ by

$$\gamma_j = \prod_{\ell \neq j} \frac{(u_\ell u_j - 1)}{u_j - u_\ell} \left(\frac{1 - u_j^2}{\rho(q^{-1} - q)} \prod_{\ell \neq j} u_\ell + \begin{cases} 1 & \text{if } r \text{ is odd} \\ -u_j & \text{if } r \text{ is even} \end{cases} \right) \tag{3.4}$$

The elements γ_j arise as the unique solutions to the system of linear equations:

$$\sum_j \frac{1}{1 - u_iu_j} \gamma_j = \frac{1}{1 - u_i^2} + \frac{1}{\rho(q^{-1} - q)} \quad (1 \le i \le r) \tag{3.5}$$

Then one has the following analogue of the theorem of Wilcox & Yu cited above:

Theorem 3.6 ([6], Theorem 3.10)**.** *Let S be an integral ground ring with parameters ρ, q, δ_j $(j \ge 0)$ and $u_1, \dots, u_r$. Assume that $(q - q^{-1}) \neq 0$, that the elements u_i are distinct, and that $u_iu_j \neq 1$ for all i, j. Define γ_j in the field of fractions of S by (3.4), for $1 \le j \le r$. Then the following conditions are equivalent:*

1. *$\{e_1, y_1e_1, \dots, y_1^{r-1}e_1\} \subseteq W_{2,S}$ is linearly independent over S.*
2. *For all $a \ge 0$, we have $\delta_a = \sum_{j=1}^r \gamma_j u_j^a$.*

Of course, by Theorem 3.3, the conditions are equivalent to the admissibility of S (in the special case considered, namely that the u_i are distinct and $u_iu_j \neq 1$ or all i, j.)

Although the γ_j are rational functions with singularities at $u_i = u_j$, one can show that the quantities $(q - q^{-1})\sum_{j=1}^r \gamma_j u_j^a$ are polynomials in $u_1, \dots, u_r$, $\rho^{\pm 1}$, and $(q - q^{-1})$, as follows: Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, $\boldsymbol{\rho}$, $\boldsymbol{q}$, and t be algebraically independent indeterminants over $\mathbb{Z}$. Define

$$G(t) = G(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r; t) = \prod_{\ell=1}^r \frac{t - \boldsymbol{u}_\ell}{t\boldsymbol{u}_\ell - 1}. \tag{3.6}$$

Let $\mu_a = \mu_a(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$ denote the a–th coefficient of the formal power series expansion of $G(t)$. Notice that each μ_a is a symmetric polynomial in $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$ and that $G(t^{-1}) = G(t)^{-1}$. Define

$$Z(t) = Z(t; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{\rho}, \boldsymbol{q}) = -\boldsymbol{\rho}^{-1} + (\boldsymbol{q} - \boldsymbol{q}^{-1}) \frac{t^2}{t^2 - 1} + A(t)\, G(t^{-1}), \tag{3.7}$$

where

$$A(t) = \begin{cases} -\rho^{-1}a_0 + (\boldsymbol{q} - \boldsymbol{q}^{-1})t/(t^2 - 1) & \text{if } r \text{ is odd, and} \\ \rho^{-1}a_0 - (\boldsymbol{q} - \boldsymbol{q}^{-1})t^2/(t^2 - 1) & \text{if } r \text{ is even.} \end{cases}$$

In the following, we use the notation $\delta_{(P)} = 1$ if (P) is true and $\delta_{(P)} = 0$ if (P) is false. Write a_j for $a_j(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r) = (-1)^{r-j}\varepsilon_j(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r)$.

Lemma 3.7 ([6], Lemma 3.18; [9], Lemmas 2.23 and 2.28)**.**
Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, $\boldsymbol{\rho}$, $\boldsymbol{q}$, and t be algebraically independent indeterminants over $\mathbb{Z}$. Define $\eta_a = \sum_j \gamma_j \boldsymbol{u}_j^a$ for $a \geq 0$, where γ_j is given by (3.4). Then

(1) $(\boldsymbol{q} - \boldsymbol{q}^{-1}) \sum_{a\geq 0} \eta_a t^{-a} = Z(t; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{\rho}, \boldsymbol{q})$.

Now let R be any commutative ring with invertible elements ρ, q, and $u_1, \dots, u_r$, and additional elements η_a, $a \geq 0$. Let t be an indeterminant over R. Let $\mu_a = \mu_a(u_1, \dots, u_r)$ be the coefficients of the formal power series expansion of $G(u_1, \dots, u_r; t)$. Suppose that

$$(q - q^{-1}) \sum_{a\geq 0} \eta_a t^{-a} = Z(t; u_1, \dots, u_r, \rho, q).$$

Then

(2) *If r is odd, then for $a \geq 0$,*

$$\begin{aligned}(q - q^{-1})\eta_a = -\delta_{(a=0)}\, \rho^{-1} + (q - q^{-1})\delta_{(a \text{ is even})} \\ - \mu_a \rho^{-1} a_0 + (q - q^{-1})(\mu_{a-1} + \mu_{a-3} + \cdots).\end{aligned}$$

(3) *If r is even, then for $a \geq 0$,*

$$\begin{aligned}(q - q^{-1})\eta_a = -\delta_{(a=0)}\, \rho^{-1} + (q - q^{-1})\delta_{(a \text{ is even})} \\ + \mu_a \rho^{-1} a_0 - (q - q^{-1})(\mu_a + \mu_{a-2} + \mu_{a-4} + \cdots).\end{aligned}$$

(4) $(q - q^{-1})\eta_0 = (a_0^2 - 1)\rho^{-1} + (q - q^{-1})(1 - \delta_{(r \text{ is even})}\, a_0)$.

(5) *For all $a \geq 0$, $(q - q^{-1})\eta_a$ is an element of the ring $\mathbb{Z}[u_1, \dots, u_r, q - q^{-1}, \rho^{-1}]$, and is symmetric in $u_1, \dots, u_r$.*

Remark 3.8. In the lemma, it is not assumed that we are working in a ground ring, i.e. that condition (2.1) holds.

Suppose that S is an integral ground ring in which the u_j are distinct, $u_i u_j \neq 1$ for all i, j, and $q - q^{-1} \neq 0$. Suppose, moreover, that S is admissible, that is $\{e, y_1 e, \dots, y_1^{r-1} e\} \subseteq W_{2,S}$ is linearly independent over S. Then by Theorem 3.6, we have $\delta_a = \sum_{j=1}^r \gamma_j u_j^a$ for $a \geq 0$. It then follows from Lemma 3.7, part (1), that

$$(q - q^{-1}) \sum_{a\geq 0} \delta_a t^{-a} = Z(t; u_1, \dots, u_r, \rho, q). \tag{3.8}$$

However, Equation (3.8) makes sense as a condition on ground rings, without any special assumptions on the elements u_i; this motivates the following definition of Rui and Xu:

Definition 3.9 (Rui and Xu, [9])**.** Let S be a ground ring with parameters ρ, q, δ_j $(j \geq 0)$ and $u_1, \dots, u_r$. Assume that $(q - q^{-1})$ is not a zero–divisor in S. One says that S is u–*admissible* (or that the parameters are u–*admissible*) if

$$(q - q^{-1}) \sum_{a \geq 0} \delta_a t^{-a} = Z(t; u_1, \dots, u_r, \rho, q),$$

where Z is defined in Equation (3.7).

Remark 3.10.

(1) Suppose that S is a u–admissible ground ring. Then conclusions (2)–(5) of Lemma 3.7 hold, with η_a replaced with δ_a. Moreover, statement (4) of Lemma 3.7 together with the ground ring condition (2.1) implies that condition (3.2) holds. If, in addition, R is assumed to be integral, then we have $\rho = \pm a_0$ if r is odd, and $\rho \in \{q^{-1}a_0, -qa_0\}$ if r is even.

(2) Let S be a ground ring with admissible (resp. u–admissible) parameters ρ, q, δ_j $(j \geq 0)$, and $u_1, \dots, u_r$. Then

$$\rho, -q^{-1}, \delta_j \ (j \geq 0), \text{ and } u_1, \dots, u_r$$

and

$$-\rho, -q, \delta_j \ (j \geq 0), \text{ and } u_1, \dots, u_r$$

are also sets of admissible (resp. u–admissible) parameters. If S is a ground ring with admissible (resp. u–admissible) parameters and $\varphi : S \to S'$ is a morphism of ground rings in the sense of Definition 2.4, such that $\varphi(q - q^{-1})$ is not a zero–divisor, then S' is also admissible (resp. u–admissible).

(3) Considering parts (1) and (2) of this remark, if S is a u–admissible integral ground ring, one can assume $\rho = -a_0 = \prod_{j=1}^r u_j$, if r is odd, and $\rho = q^{-1}a_0 = q^{-1} \prod_{j=1}^r u_j$, if r is even.

4. Equivalence of admissibility and u–admissibility

Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_r$, $\boldsymbol{\rho}$, $\boldsymbol{q}$, and t be algebraically independent indeterminants over $\mathbb{Z}$. Define $Z(t) \in \mathbb{Q}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{\rho}, \boldsymbol{q}, t)$ by Equation (3.7), and define η_a for $a \geq 0$ by

$$(\boldsymbol{q} - \boldsymbol{q}^{-1}) \sum_{a \geq 0} \eta_a t^{-a} = Z(t; \boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{\rho}, \boldsymbol{q}).$$

Then statements (2)–(5) of Lemma 3.7 hold; in particular, by part (5) of Lemma 3.7, $(\boldsymbol{q} - \boldsymbol{q}^{-1})\eta_a \in \mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{q} - \boldsymbol{q}^{-1}, \boldsymbol{\rho}^{-1}]$.

Lemma 4.1. *The elements η_j satisfy*

$$\begin{aligned} &[\boldsymbol{\rho}^{-1} a_0 - \delta_{(r \text{ is even})}(\boldsymbol{q} - \boldsymbol{q}^{-1})](a_0 a_\ell - a_{r-\ell}) \\ &+ (\boldsymbol{q} - \boldsymbol{q}^{-1}) \left(\sum_{j=1}^{r-\ell} \eta_j a_{j+\ell} - \sum_{j=\max(\ell+1, \lceil r/2 \rceil)}^{\lfloor (\ell+r)/2 \rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2 \rceil}^{\min(\ell, \lceil r/2 \rceil - 1)} a_{2j-\ell} \right) = 0, \end{aligned} \tag{4.1}$$

for $1 \leq \ell \leq r - 1$.

Proof. We have

$$(\boldsymbol{q}-\boldsymbol{q}^{-1})\sum_{a\ge 0}\eta_a t^a = Z(t^{-1}) \tag{4.2}$$
$$= -\boldsymbol{\rho}^{-1} + (\boldsymbol{q}-\boldsymbol{q}^{-1})/(1-t^2) + A(t^{-1})\prod_{\ell=1}^{r}\frac{t-\boldsymbol{u}_\ell}{t\boldsymbol{u}_\ell - 1}.$$

Multiplying both sides of (4.2) by $\prod_{\ell=1}^{r}(t\boldsymbol{u}_\ell - 1)$ gives

$$(\boldsymbol{q}-\boldsymbol{q}^{-1})\prod_{\ell=1}^{r}(t\boldsymbol{u}_\ell - 1)\sum_{a\ge 0}\eta_a t^a \tag{4.3}$$
$$= -\boldsymbol{\rho}^{-1}\prod_{\ell=1}^{r}(t\boldsymbol{u}_\ell - 1) + \frac{\boldsymbol{q}-\boldsymbol{q}^{-1}}{1-t^2}\prod_{\ell=1}^{r}(t\boldsymbol{u}_\ell - 1) + A(t^{-1})\prod_{\ell=1}^{r}(t-\boldsymbol{u}_\ell).$$

For $1\le \ell\le r-1$, the coefficient of $t^{r-\ell}$ on the left side of (4.3) is

$$(-1)^r\,(\boldsymbol{q}-\boldsymbol{q}^{-1})\left(\eta_0 a_\ell + \sum_{j=1}^{r-\ell}\eta_j a_{j+\ell}\right). \tag{4.4}$$

Taking into account the formula for η_0 in part (4) of Lemma 3.7, (4.4) becomes

$$(-1)^r\Bigg((a_0^2-1)\boldsymbol{\rho}^{-1}a_\ell + (\boldsymbol{q}-\boldsymbol{q}^{-1})a_\ell \tag{4.5}$$
$$- \delta_{(r\text{ is even})}(\boldsymbol{q}-\boldsymbol{q}^{-1})a_0 a_\ell + (\boldsymbol{q}-\boldsymbol{q}^{-1})\sum_{j=1}^{r-\ell}\eta_j a_{j+\ell}\Bigg).$$

Now suppose that r is odd. Then for $1\le \ell\le r-1$, the coefficient of $t^{r-\ell}$ on the right side of (4.3) is

$$\boldsymbol{\rho}^{-1}a_\ell - \boldsymbol{\rho}^{-1}a_0 a_{r-\ell} \tag{4.6}$$
$$-(\boldsymbol{q}-\boldsymbol{q}^{-1})\sum_{i=0}^{\lfloor (r-\ell)/2\rfloor} a_{\ell+2i} + (\boldsymbol{q}-\boldsymbol{q}^{-1})\sum_{i=0}^{\lfloor (r-1-\ell)/2\rfloor} a_{r-1-\ell-2i}.$$

Continuing with the case that r is odd, and setting (4.5) equal to (4.6), we get

$$0 = \boldsymbol{\rho}^{-1}a_0(a_0 a_\ell - a_{r-\ell}) \tag{4.7}$$
$$+(\boldsymbol{q}-\boldsymbol{q}^{-1})\left(\sum_{j=1}^{r-\ell}\eta_j a_{j+\ell} + a_\ell - \sum_{i=0}^{\lfloor (r-\ell)/2\rfloor} a_{\ell+2i} + \sum_{i=0}^{\lfloor (r-1-\ell)/2\rfloor} a_{r-i-\ell-2i}\right).$$

By examining cases, according to the parity of ℓ and the sign of $\ell+1-\lceil r/2\rceil$, one can check that the expression on the second line of (4.7) is equal to

$$(\boldsymbol{q}-\boldsymbol{q}^{-1})\left(\sum_{j=1}^{r-\ell}\eta_j a_{j+\ell} - \sum_{j=\max(\ell+1,\lceil r/2\rceil)}^{\lfloor (\ell+r)/2\rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2\rceil}^{\min(\ell,\lceil r/2\rceil-1)} a_{2j-\ell}\right). \tag{4.8}$$

For example, if ℓ is odd and $\ell+1 \le \lceil r/2 \rceil = (r+1)/2$, then

$$\begin{aligned}
(4.9)\qquad & -\sum_{j=\max(\ell+1,\lceil r/2\rceil)}^{\lfloor(\ell+r)/2\rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2\rceil}^{\min(\ell,\lceil r/2\rceil-1)} a_{2j-\ell} \\
&= \sum\{a_k | k \text{ odd and } 1 \le k \le \ell\} - \sum\{a_k | k \text{ odd and } r+1-\ell \le k \le r\},
\end{aligned}$$

while

$$\begin{aligned}
(4.10)\qquad & a_\ell - \sum_{i=0}^{\lfloor(r-\ell)/2\rfloor} a_{\ell+2i} + \sum_{i=0}^{\lfloor(r-1-\ell)/2\rfloor} a_{r-i-\ell-2i} \\
&= \sum\{a_k | k \text{ odd and } 1 \le k \le r-1-\ell\} - \sum\{a_k | k \text{ odd and } \ell+2 \le k \le r\}.
\end{aligned}$$

The summands $\{a_k | k \text{ odd and } \ell+2 \le k \le r-1-\ell\}$ appear in both of the sums on the last line, so they cancel to give

$$\begin{aligned}
(4.11)\qquad & a_\ell - \sum_{i=0}^{\lfloor(r-\ell)/2\rfloor} a_{\ell+2i} + \sum_{i=0}^{\lfloor(r-1-\ell)/2\rfloor} a_{r-i-\ell-2i} \\
&= \sum\{a_k | k \text{ odd and } 1 \le k \le \ell\} - \sum\{a_k | k \text{ odd and } r+1-\ell \le k \le r\}.
\end{aligned}$$

Comparing (4.9) and (4.11) gives

$$\begin{aligned}
(4.12)\qquad & a_\ell - \sum_{i=0}^{\lfloor(r-\ell)/2\rfloor} a_{\ell+2i} + \sum_{i=0}^{\lfloor(r-1-\ell)/2\rfloor} a_{r-i-\ell-2i} \\
&\qquad = -\sum_{j=\max(\ell+1,\lceil r/2\rceil)}^{\lfloor(\ell+r)/2\rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2\rceil}^{\min(\ell,\lceil r/2\rceil-1)} a_{2j-\ell},
\end{aligned}$$

and therefore the second line of (4.7) is equal to(4.8). The other cases are handled similarly. This completes the proof of the lemma when r is odd.

Now consider the case that r is even. Then for $1 \le \ell \le r-1$, the coefficient of $t^{r-\ell}$ on the right side of (4.3) is

$$\begin{aligned}
(4.13)\qquad & -\boldsymbol{\rho}^{-1} a_\ell + \boldsymbol{\rho}^{-1} a_0 a_{r-\ell} \\
&\qquad + (\boldsymbol{q} - \boldsymbol{q}^{-1}) \sum_{i=0}^{\lfloor(r-\ell)/2\rfloor} a_{\ell+2i} - (\boldsymbol{q} - \boldsymbol{q}^{-1}) \sum_{i=0}^{\lfloor(r-\ell)/2\rfloor} a_{r-\ell-2i}.
\end{aligned}$$

Setting (4.5) equal to (4.13), we get

$$
\begin{aligned}
0 =& \boldsymbol{\rho}^{-1} a_0^2 a_\ell - \boldsymbol{\rho}^{-1} a_0 a_{r-\ell} - (\boldsymbol{q} - \boldsymbol{q}^{-1}) a_0 a_\ell \\
&+ (\boldsymbol{q} - \boldsymbol{q}^{-1}) \left(\sum_{j=1}^{r-\ell} \eta_j a_{j+\ell} + a_\ell - \sum_{i=0}^{\lfloor (r-\ell)/2 \rfloor} a_{\ell+2i} + \sum_{i=0}^{\lfloor (r-\ell)/2 \rfloor} a_{r-\ell-2i} \right) \\
=& \left[\boldsymbol{\rho}^{-1} a_0 - (\boldsymbol{q} - \boldsymbol{q}^{-1}) \right] (a_0 a_\ell - a_{r-\ell}) \\
&+ (\boldsymbol{q} - \boldsymbol{q}^{-1}) \left(\sum_{j=1}^{r-\ell} \eta_j a_{j+\ell} + a_\ell - a_{r-\ell} - \sum_{i=0}^{\lfloor (r-\ell)/2 \rfloor} a_{\ell+2i} + \sum_{i=0}^{\lfloor (r-\ell)/2 \rfloor} a_{r-\ell-2i} \right).
\end{aligned} \tag{4.14}
$$

As in the case that r is odd, one can show that the expression in the last line of (4.14) is equal to (4.8). This completes the proof in case r is even. □

Corollary 4.2. *Let*

$$
\Lambda = \mathbb{Z}[\boldsymbol{u}_1^{\pm 1}, \dots, \boldsymbol{u}_r^{\pm 1}, \boldsymbol{\rho}^{\pm 1}, \boldsymbol{q}^{\pm 1}, (\boldsymbol{q} - \boldsymbol{q}^{-1})^{-1}]/I
$$

where I is the ideal generated by $\boldsymbol{\rho}^{-1} a_0 - \boldsymbol{\rho} a_0^{-1} - \delta_{(r \text{ is even})}(\boldsymbol{q} - \boldsymbol{q}^{-1})$. The image of the elements η_j in Λ satisfy

$$
\begin{aligned}
&\boldsymbol{\rho}(a_\ell - a_{r-\ell}/a_0) \\
&\qquad + (\boldsymbol{q} - \boldsymbol{q}^{-1}) \left(\sum_{j=1}^{r-\ell} \eta_j a_{j+\ell} - \sum_{j=\max(\ell+1, \lceil r/2 \rceil)}^{\lfloor (\ell+r)/2 \rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2 \rceil}^{\min(\ell, \lceil r/2 \rceil - 1)} a_{2j-\ell} \right) = 0,
\end{aligned} \tag{4.15}
$$

for $1 \le \ell \le r-1$.

Lemma 4.3. *For $m \ge r$, one has $\sum_{j=0}^r a_j \eta_{j+m-r} = 0$.*

Proof. For $m \ge r$, the coefficient of t^m on the left side of (4.3) is

$$
(-1)^r (\boldsymbol{q} - \boldsymbol{q}^{-1}) \sum_{j=0}^r a_j \eta_{j+m-r}.
$$

Thus, we have to show that the coefficient of t^m on the right side of (4.3) is zero.

If r is odd, then the right side of (4.3) is

$$
\begin{aligned}
&-\boldsymbol{\rho}^{-1} \prod_{\ell=1}^r (t u_\ell - 1) - \boldsymbol{\rho}^{-1} a_0 \prod_{\ell=1}^r (t - u_\ell) \\
&\qquad + (\boldsymbol{q} - \boldsymbol{q}^{-1}) \left(\frac{\prod_{\ell=1}^r (t u_\ell - 1)}{1 - t^2} + \frac{t \prod_{\ell=1}^r (t - u_\ell)}{1 - t^2} \right)
\end{aligned} \tag{4.16}
$$

For $m > 0$, the coefficient of t^m in the first line of (4.16) is zero. Moreover, the coefficient of t^r is $-\boldsymbol{\rho}^{-1}(\prod_{\ell=1}^r \boldsymbol{u}_\ell - a_0) = 0$.

Write $a_k = 0$ if $k < 0$ or $k > r$. Then the second line of (4.16) expands to

$$(4.17)\quad \begin{aligned} &(\boldsymbol{q}-\boldsymbol{q}^{-1})\left((-1)^r\left[\sum_{j=0}^{r} t^j a_{r-j}\right]\left[\sum_{\ell\geq 0} t^{2\ell}\right]+\left[\sum_{j=0}^{r} t^j a_j\right]\left[\sum_{\ell\geq 0} t^{2\ell+1}\right]\right)\\ &=(\boldsymbol{q}-\boldsymbol{q}^{-1})\left((-1)^r\sum_{m\geq 0}\left[\sum_{\ell\geq 0} a_{r-m+2\ell}\right]t^m+\sum_{m\geq 0}\left[\sum_{\ell\geq 0} a_{m-1-2\ell}\right]t^m\right)\end{aligned}$$

For $m \geq r$, the coefficient of t^m in (4.17) is zero. Thus, for $m \geq r$, the coefficient of t^m in (4.16) is zero.

The proof when r is even is similar. □

Theorem 4.4. *Let S be a ground ring with parameters ρ, q, δ_j and $u_1, \dots, u_r$, with $q - q^{-1}$ not a zero–divisor. Then S is admissible if and only if S is u–admissible.*

Proof. Let η_a $(a \geq 0)$ be determined by

$$(q-q^{-1})\sum_{a\geq 0}\eta_a t^{-a} = Z(t; u_1, \dots, u_r, \rho, q),$$

Suppose that the parameters are u–admissible. Then $\delta_a = \eta_a$ for $a \geq 0$ by definition of u–admissibility, and the assumption on $q - q^{-1}$. Condition (3.2) holds by Remark 3.10, part (1), and because of this, it follows from Corollary 4.2 that the parameters satisfy condition (3.1). Moreover, the parameters satisfy condition (3.3) according to Lemma 4.3. Thus the parameters are admissible.

Conversely, suppose that the parameters are admissible. The admissibility conditions (3.1) and (3.3) and the ground ring condition (2.1) uniquely determine the quantities $(q - q^{-1})\delta_a$ for $(a \geq 0)$ as Laurent polynomials in ρ and $u_1, \dots, u_r$. Indeed, note that (3.1) is a system of linear equations in the variables $(q - q^{-1})\delta_j$ $(1 \leq j \leq r-1)$ with unitriangular matrix of coefficients. (Compare [6], Remark 3.7.) For $a \geq r$, the weak admissibility condition (3.3), determines δ_a as a polynomial in $u_1, \dots, u_r$ and $\{\delta_j : j < a\}$. Finally (2.1) determines $(q - q^{-1})\delta_0$.

Consider the new set of parameters $P' = (\rho, q, \eta_a, u_1, \dots, u_r)$ with the δ_a's replaced by the η_a's and the other parameters unchanged. We claim that P' is also a set of admissible parameters (satisfying the ground ring condition). In fact, condition (3.2) holds for P', because it involves only ρ, q and $u_1, \dots, u_r$. The ground ring condition (2.1) for P' follows from condition (3.2) and Lemma 3.7 part (4). P' satisfies conditions (3.1) and (3.3) by Corollary 4.2 and Lemma 4.3. This finishes the verification of the claim.

Since, P' is a set of admissible parameters, the quantities $(q - q^{-1})\eta_a$ for $(a \geq 0)$ are given by the same Laurent polynomials in the remaining parameters as are the quantities $(q - q^{-1})\delta_a$ for $(a \geq 0)$. Since $q - q^{-1}$ is not a zero divisor, we have $\delta_a = \eta_a$ for all $a \geq 0$, and hence the original parameters are u–admissible. □

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